inequalities dec 2025 lecture

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Problems with a ($\star$) are particularly nice; problems with a ($\dagger$) are particularly hard (like IMO 2.5 to 3 level).

1 Inequality techniques

- AM-GM, AM-QM, AM-HM
- Weighted AM-GM (your best friend when the inequality looks weird!!)
- Cauchy-Schwarz
- Holder (basically weighted Cauchy-Schwarz)
- Rearrangement inequality
- Substitutions to make expressions homogenous (eg if $xyz = 1$, can sub $x, y, z = \frac{a}{b}, \frac{b}{c}, \frac{c}{a}$) or nicer (eg let $d_i = x_{i+1} - x_i$)

2 Going back to basics

Unfortunately these days most olympiad inequalities are designed so you can't just throw techniques at them and win.

The main idea of this lecture is

- Looking at inequalities as functions in one or more variables: the statement $LHS \geq RHS$ can be converted to $f(x_1, \ldots x_n) = LHS - RHS \geq 0$.
- Then, finding where this f takes its minimum values, and then checking that they are nonnegative.

In general, try to approach inequalities like any other olympiad problem:

- When do we have equality (or when are the two sides closest together)?
- If I fix all variables except one, and vary the last one, what happens - how do I minimise the gap?
- Can you induct? (i.e. if adding a term doesn't significantly disturb the expression)
- Can you simplify the expression (eg telescoping, or substituting a different set of variables)?

3 Equality cases

Looking at equality cases can give you a clue about which of the standard inequality techniques will apply here, or perhaps what expressions you want to create. For example, suppose your solution is a chain of inequalities ($LHS \geq \ldots \geq RHS$) one of which is true by AM-GM. Then, when LHS = RHS, all the steps in the middle become equalities, so your equality case for the problem must also be an equality case for the AM-GM you used. This can be used to reverse-engineer steps.

Note: all power mean inequalities have equality case when all the terms are equal. Cauchy-Schwarz, however, has equality case when the ratios of a_i to b_i are equal, in

$$(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \geq (a_1 b_1, \ldots, a_n b_n)^2.$$

Weighted AM-GM is REALLY GOOD - a useful form of it is

$$\frac{x_1 + x_2 + \cdots + x_n}{w_1 + \ldots + w_n} \geq \sqrt[w_1+\ldots+w_n]{\left(\frac{x_1}{w_1}\right)^{w_1}\left(\frac{x_2}{w_2}\right)^{w_2}\cdots\left(\frac{x_n}{w_n}\right)^{w_n}}$$

with equality when all the x_i/w_i are equal.

3.1 Problems:

1. ($\star$) Let a, b, c, d be positive real numbers satisfying $(a+c)(b+d) = ac + bd$. Find the smallest possible value of
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$
2. ($\star$) Let the real numbers a, b, c, d satisfy $a + b + c + d = 6$ and $a^2 + b^2 + c^2 + d^2 = 12$. Prove that
$$36 \leq 4(a^3 + b^3 + c^3 + d^3) - (a^4 + b^4 + c^4 + d^4) \leq 48.$$
3. Prove that for $a, b, c > 0$ we have
$$\sum_{sym}(a^4b^2 + 2a^3b^2c) \geq \sum_{sym}(a^4bc + a^3b^3 + a^2b^2c^2).$$
4. Find all lists $(x_1, x_2, \ldots, x_{2020})$ of non-negative real numbers such that the following three conditions are all satisfied: $x_1 \leq x_2 \leq \ldots \leq x_{2020}$; $x_{2020} \leq x_1 + 1$; there is a permutation $(y_1, y_2, \ldots, y_{2020})$ of $(x_1, x_2, \ldots, x_{2020})$ such that
$$\sum_{i=1}^{2020}((x_i + 1)(y_i + 1))^2 = 8\sum_{i=1}^{2020} x_i^3.$$
5. For $n \geq 2$, let $a_1, \ldots, a_n$ be positive real numbers such that
$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \leq \left(n + \frac{1}{2}\right)^2.$$
Prove that $\max(a_1, a_2, \ldots, a_n) \leq 4\min(a_1, a_2, \ldots, a_n)$
6. ($\star$) Find all real numbers x, y, z which satisfy
$$x^3 = 3x - 12y + 50,$$
$$y^3 = 12y + 3z - 2,$$
$$z^3 = 27z + 27x.$$
7. ($\dagger$) (Russian magic) Find the maximal value of
$$S = \sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{c+7}} + \sqrt[3]{\frac{c}{d+7}} + \sqrt[3]{\frac{d}{a+7}}$$
where a, b, c, d are nonnegative real numbers which satisfy $a + b + c + d = 100$.

4 Perturbation

By taking a multivariable expression and keeping every variable constant except for one, we can simplify it into one of the nice single-variable functions we're used to. Calculus is our best tool for finding the minima and maxima of such functions, but often perturbation can be done without it. Sometimes if there is a constraint, you will have to vary two or more variables at once so that the constraint is kept.

Special cases of perturbation are **convexity** and **linearity**:

- A convex function is maximised at the endpoints of the domain; a concave function is minimised at the endpoints. This is essentially what Jensen says, but it can be applied more generally.
- The sum of multiple convex functions is convex, and likewise with concave functions.
- A special case (in some sense) is a linear function, which is maximised at the endpoints of the domain.

4.1 Problems:

1. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers, where $n \geq 2$. Let K, L be the maximum and minimum, respectively, of $b_1, b_2, \ldots, b_n$. Prove that
$$\sum_{i<j} a_i a_j |b_i - b_j| \leq \frac{1}{2}(K - L)(a_1 + a_2 + \ldots + a_n)^2.$$

2. ($\star$) If $0 \leq a_1, \ldots, a_n \leq 1$, show
$$\frac{1 - a_1 a_2 \ldots a_n}{n} \leq \frac{1}{1 + a_1 + \ldots + a_n}$$

3. (Bashy) Maximise
$$\sum_{1 \leq r < s \leq 2n} (s - r - n) x_r x_s$$
for a fixed positive integer n, given $-1 \leq x_i \leq 1$ for each i.

4. (†)
$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i - x_j|} \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i + x_j|}$$

5. (Bashy) Find the maximal value of
$$S = \sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{c+7}} + \sqrt[3]{\frac{c}{d+7}} + \sqrt[3]{\frac{d}{a+7}}$$
where a, b, c, d are nonnegative real numbers which satisfy $a + b + c + d = 100$.

5 Telescoping:

Basically rewriting terms - usually using substitutions to make expressions nicer - so that almost all of them cancel.

5.1 Problems:

1. ($\star$) Let $a_1, \ldots, a_i \ldots$ be a sequence of positive real numbers such that

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + k - 1}$$

for all $k \geq 1$. Prove that $a_1 + \ldots + a_n \geq n$ for all $n \geq 2$.

2. ($\star$) Let $a_2, \ldots, a_n$ be $n-1$ positive real numbers, where $n \geq 3$, such that $a_2 a_3 \ldots a_n = 1$.. Prove that

$$(1 + a_2)^2 \ldots (1 + a_n)^n > n^n.$$

3. ($\star$) Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers. Prove that

$$\frac{1}{1+a_1} + \frac{a_1}{(1+a_1)(1+a_2)} + \frac{a_1 a_2}{(1+a_1)(1+a_2)(1+a_3)} + \cdots + \frac{a_1 a_2 \cdots a_{n-1}}{(1+a_1)(1+a_2)\cdots(1+a_n)} \leq 1.$$

4. Let n be a positive integer and $a_1, a2, \ldots, a_n$ be positive real numbers. Prove that

$$\sum_{i=1}^{n} \frac{1}{2^i} (\frac{2}{1+a_i})^{2^i} \geq \frac{2}{1 + a_1 a_2 \ldots a_n} - \frac{1}{2^n}.$$

5. ($\dagger$) Let $a_1, \ldots, a_n$ be positive real numbers with $a_1 + \ldots + a_n = 1$ and $n \geq 2$. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{1 - a_k} (a_1 + \ldots + a_{k-1})^2 < \frac{1}{3}.$$

6. ($\dagger$) Determine the largest real number a such that for all $n \geq 1$ and for all real numbers $x_0, x_1, \ldots, x_n$ satisfying $0 = x_0 < x_1 < \ldots < x_n$, we have

$$\frac{1}{x_1 - x_0} + \ldots + \frac{1}{x_n - x_{n-1}} \geq a \left(\frac{2}{x_1} + \frac{3}{x_2} + \ldots + \frac{n+1}{x_n} \right).$$

6 Induction

6.1 Problems:

1. ($\star$) (uses perturbation) For nonnegative reals $x_1, \ldots, x_n$, show that

$$\frac{(x_1 + \ldots + x_n)^2}{4} \geq x_1 x_2 + \ldots + x_{n-1} x_n.$$

2. ($\star$) Let $n \geq 3$ be an integer and $t_1, t_2, \ldots, t_n$ be positive real numbers such that

$$n^2 + 1 > (t_1 + \ldots + t_n)(\frac{1}{t_1} + \ldots + \frac{1}{t_n}).$$

Show that t_i, t_j, t_k are the sides of a triangle for all $1 \leq i < j < k \leq n$.

3. For $n \geq 2$ let $a_1, a_2, \ldots a_n$ be positive real numbers such that

$$(a_1 + a_2 + \cdots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \leq \left(n + \frac{1}{2} \right)^2.$$

Prove that $\max(a_1, a_2, \ldots, a_n) \leq 4 \min(a_1, a_2, \ldots, a_n)$.

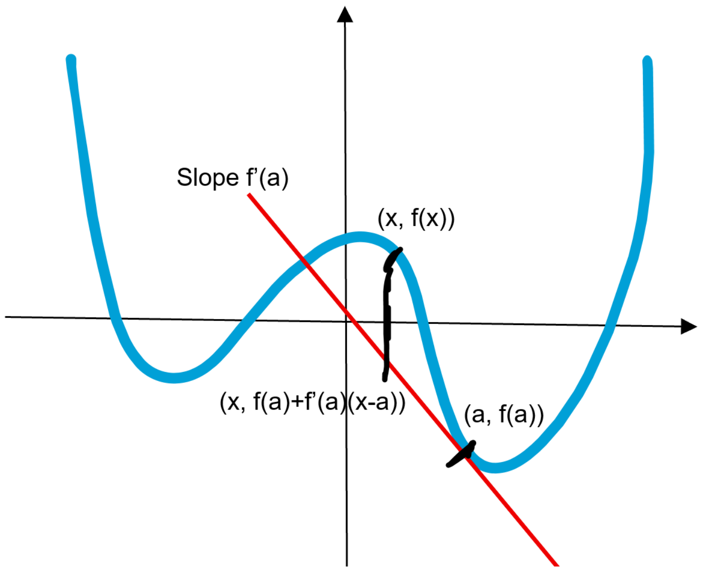
7 Calculus estimation methods

Tangent line trick:
Suppose you have an inequality in the form $\sum f(x_i)$, i.e. a sum of functions each in one variable. If you can prove $f(x)$ lies above a certain line for all x in your domain, you can then bound each $f(x_i)$ below using that linear function, which is much easier to evaluate.

In particular, if you take the line to be the tangent to the graph of $f(x)$ at a point, what you need to prove is

$$f(x) \geq f(a) + f'(a)(x - a).$$

It similarly works if you consider a tangent line above the graph to create an upper bound.



If a function is convex then it always lies above the tangent to it at any point; however, this also happens sometimes for non-convex functions, if you restrict the domain or pick a special point.

7.1 Problems:

1. ($\star$) Prove the power means inequality using differentiation: for real numbers $x \geq y$,

$$\left(\frac{a_1^x + \ldots + a_n^x}{n}\right)^{\frac{1}{x}} \geq \left(\frac{a_1^y + \ldots + a_n^y}{n}\right)^{\frac{1}{y}}.$$

2. Let $a, b, c, d > 0$ and $a + b + c + d = 1$. Prove that

$$6(a^3 + b^3 + c^3 + d^3) \geq (a_2 + b_2 + c_2 + d_2) + \frac{1}{8}.$$

3. ($\star$) Let a, b, c be positive reals. PRove that

$$(\frac{2a}{b+c})^{\frac{2}{3}} + (\frac{2a}{b+c})^{\frac{2}{3}} + (\frac{2a}{b+c})^{\frac{2}{3}} \geq 3$$

4. ($\star$) Given positive reals a, b, c, d with $a + b + c + d = 4$, prove

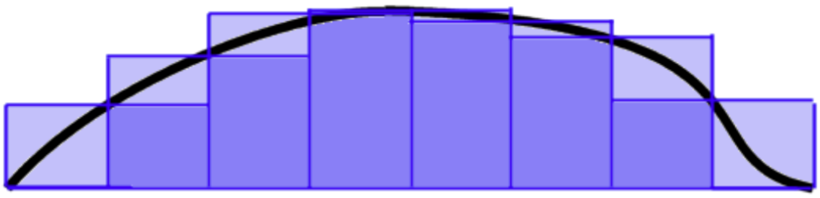
$$\sum_{cyc} \frac{a}{b^3 + 4} \geq \frac{2}{3}.$$

5. Prove that if $a+b+c=3$ then
$$\sum_{cyc} \frac{a}{2a^2+a+1} \leq \frac{3}{4}.$$

6. Let $a, b, c, d \geq -1$, $a+b+c+d=4$. Prove that
$$\frac{5-3a}{(a-3)^2} + \frac{5-3b}{(b-3)^2} + \frac{5-3c}{(c-3)^2} + \frac{5-3d}{(d-3)^2} \leq 2.$$

7.2 (†) Extra: Integration intuition

This is the idea that $\sum_{1<=i<=n} f(i)$ is approximately the integral of $f(i)$ from 1 to $n+1$ (or 0 to n if you wish). This can be helpful for finding equality cases. More specifically, if the function is convex or concave, you can get upper and lower bounds:



7.3 Problems:

1. (†) Let $a_1, \ldots, a_n$ be positive real numbers with $a_1 + \ldots + a_n = 1$ and $n \geq 2$. Prove that
$$\sum_{k=1}^{n} \frac{a_k}{1-a_k}(a_1 + \ldots + a_{k-1})^2 < \frac{1}{3}.$$

2. (†) Determine the largest real number a such that for all $n \geq 1$ and for all real numbers $x_0, x_1, \ldots, x_n$ satisfying $0 = x_0 < x_1 < \ldots < x_n$, we have
$$\frac{1}{x_1 - x_0} + \ldots + \frac{1}{x_n - x_{n-1}} \geq a(\frac{2}{x_1} + \frac{3}{x_2} + \ldots + \frac{n+1}{x_n}.$$

3. (†) Let $n \geq 1$ be an integer, and let $x_0, x_1, \ldots, x_{n+1}$ be $n+2$ non-negative real numbers that satisfy $x_i x_{i+1} - x_{i-1}^2$ for all $1 \leq i \leq n$. Show that
$$x_0 + x_1 + \ldots + x_{n+1} > \left(\frac{2n}{3}\right)^{3/2}.$$

4. (†) Let n and k be positive integers. Prove that for $a_1, \ldots, a_n \in [1, 2^k]$ one has
$$\sum_{i=1}^{n} \frac{a_i}{\sqrt{a_1^2 + \ldots + a_i^2}} \leq 4\sqrt{kn}.$$

8 Lagrange Multipliers

'Basically all you need for olympiad inequalities' - Fredy Yip

'The canonical way to solve inequalities in research - everything else is just pulling tricks out of a hat' - Fredy Yip

Intuition:

Suppose you have a function $f(x_1, \dots x_n)$ in multiple variables, and want to prove $f(x_1, \dots x_n) \geq 0$ for all $(x_1, \dots, x_n)$.

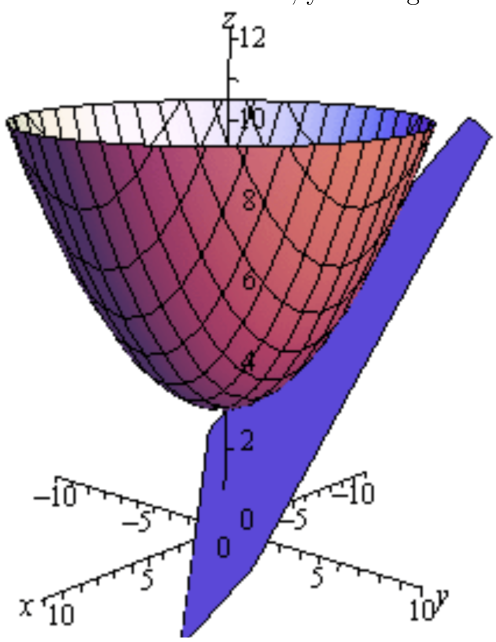
Let's visualise this in 1 variable first. We have a function $f(x)$ and want to prove it's ≥ 0. How can we do it? We could try to find all the minima by locating points where $f'(x) = 0$ and manually check them all. But it's possible that the graph becomes negative as $x \to \infty$ or $x \to -\infty$. So we also need to check that in those cases $f(x)$ remains positive. Now this is a little harder, because e.g. if you have $x^2 - 2x + 1$, how do you look at the behaviour of this as x approaches infinity?

It might be easier to do this on a bounded domain, so that x couldn't approach infinity. But even then we have problems; for example, if $f(x) = 1 - \frac{1}{x}$ is defined on $(0, 1)$, then even though there are no points where $f'(x) = 0$, it still goes to negative infinity as $x \to 0$. But the real issue here is that we have an open interval - if we had a closed interval then we simply couldn't get arbitrarily close to 0.

Therefore, we fix this issue by asking for a closed and bounded domain. 'Bounded' means the absolute value of the variable has an upper limit, and 'closed' means the domain contains its boundary. In $\mathbb{R}$, this means it's a collection of closed intervals.

Now let's consider the case where the function takes 2 variables and gives us a real number. We can visualise this as a 3D graph by plotting the points $(x, y, f(x, y))$.

Suppose you're at a point on this graphed surface and you want to get to a minimum. You can do this by always moving downwards if possible. Now if we zoom into a point on the graph enough, the graph near that point starts to look like a flat plane. If this plane is tilted, then there's a unique direction v so that if you move in the direction v, you will go directly down the plane, thus decreasing f as fast as possible.



So at any point (x, y) on the horizontal plane, we define:

$$\nabla f = \text{ the direction in which } f \text{ increases most quickly.}$$

This is also a vector in the xy-plane, and it says that if we stand at (x, y) and move a little bit in the opposite direction to ∇f, the function $f(x, y)$ will decrease.

Points on the 'tangent plane' are given by $f(x_0) + \nabla f(x_0)(x - x_0)$; notice that this looks like the equation of a line, but can be generalised to any number of dimensions.

Theorem: for $f : \mathbb{R}^n \to \mathbb{R}$,

$$\nabla f(x_1, \dots x_n) = \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}.$$

The $\frac{\partial f}{\partial x_i}$ means you take the derivative with respect to x_i while pretending all the other variables are constant.

Example: If $f(x, y, z) = x^2 + y^2 + z^2 - xy - yz - zx$, then

$$\nabla f(x, y, z) = (2x - y - z, 2y - z - x, 2z - x - y).$$

So whenever the tangent plane is tilted, we can always decrease f a little bit by modifying our input a little bit - or conversely, increase it a bit by modifying our input in the opposite direction. Therefore, **if we are at a maximum or minimum, then the tangent plane must be horizontal** - because we cannot decrease the function no matter what direction we move in. This is analogous to a single variable function having a horizontal tangent line at a point.

So it seems like if we want a maximum or minimum then we need to set $\nabla f = 0$. BUT - just as in the 1-variable case - the issue is that if your domain is unbounded, f could simply go to negative infinity as your input goes infinitely far in some direction, and we cannot detect this by looking for horizontal tangent planes. So again **we need the domain to be bounded** - and similarly to the case with $f(x) = \frac{1}{x}$ in 1 variable, we need the domain to be **closed, to contain its boundary**. We call a closed and bounded subset of $\mathbb{R}^n$ a **compact subset**.

Conversely, it is a theorem of real analysis that every function from a compact subset of $\mathbb{R}^n$ to $\mathbb{R}$ has a bounded range, and that **the range's upper and lower bounds will occur as values of the function,** either at points with gradient 0 or on the boundary of the domain. This means in particular that, to find upper or lower bounds for f, all you need to do is find the local minima and maxima, manually check them all, then manually check the values f takes on the boundary of the domain.

Now the issue with this is that usually with inequalities our domain is not bounded - for example, often it is $x_i \geq 0$ for all i. However, often the inequality is homogenous, eg $(a^2 + b^2 + c^2)^2 \geq 3(a^3 b + b^3 c + c^3 a)$. So if we **add a constraint**, say $a + b + c = 1$, and prove the inequality given the constraint, then multiplying both sides by a constant tells us the inequality is true for all (a, b, c).

So now suppose we want to maximise $f(x_1, \dots, x_n)$ with the constraint that $g(x_1, \dots, x_n) = c$ for some function g. (Note that g doesn't necessarily have to be homogenous). Suppose we are at a point $(x_1, \dots, x_n)$ corresponding to a maximum or minimum of $f(x_1, \dots, x_n)$. This means we **cannot** make f increase by moving a little bit while keeping $g(x_1, \dots x_n)$ constant.

Let us first suppose that ∇g is nonzero at this point x.

Recall that ∇f is the direction we move in to decrease f as efficiently as possible. As we attempt to decrease f, we can move in **any direction that keeps g constant**. But if it is impossible to decrease f from this point, none of these directions have **any component** in the direction of ∇f – meaning that all of them are **perpendicular to ∇f.**

But also - since any movement in the direction of ∇g increases g (and movement in the opposite direction decreases g), **the only way to keep g constant is to move perpendicular to ∇g**. Hence, all directions perpendicular to ∇g are also perpendicular to ∇f. Thus, ∇f is parallel to ∇g, so we can write $\nabla f = \lambda \nabla g$ for some real constant λ. (Note that this also works when ∇f is 0; just take $\lambda = 0$.)

But wait - what if there is no direction we can possibly move in to keep g constant, because - for example - $g(x)$ is a minimum/maximum? Then $\nabla g = 0$, and this is our second case.

Finally, if our point $(x_1, \ldots, x_n)$ is the boundary of the domain, then regardless of what direction ∇f is in, it might be impossible to move in that direction because you'd leave the domain. Therefore we also need to check the boundary, and this is our third case.

This gives **the statement of Lagrange Multipliers:**

Let $f(x), g(x)$ be defined on a compact subset of $\mathbb{R}^n$, with codomain $\mathbb{R}$. For all x such that $f(x)$ is a maximum/minimum,

- $\nabla f(x) = \lambda \nabla g(x)$ for some real constant λ
- OR $\nabla g(x) = 0$
- OR x lies on the boundary of the domain.

This idea can be extended to multiple constraint functions - if you have functions $g_1, \ldots g_k$, then

- $\nabla f(x) = \sum_{1 \le i \le k} \lambda_i \nabla g_i(x)$ for some real constants $\lambda_1, \ldots, \lambda_k$ - that is, $\nabla f(x)$ is a linear combination of the $\nabla g_i(x)$
- OR one or more of the $\nabla g_i(x) = 0$
- OR x lies on the boundary of the domain.

But this is not very useful for olympiad and more details can be found e.g. on Wikipedia.

Lagrange multipliers is an excellent tool for homogenous inequalities, but it is also very useful whenever you have a weird non-homogenous condition, such as $ab + bc + ca = a + b + c$.

8.1 Problems:

1. Prove AM-GM using Lagrange multipliers.
2. Prove Cauchy-Schwarz using Lagrange multipliers
3. For nonnegative reals $x_1, \ldots, x_n$, show that
$$\frac{(x_1 + \ldots + x_n)^2}{4} \ge x_1 x_2 + \ldots + x_{n-1} x_n.$$
4. Let $a, b, c \ge -0$ and $a + b + c = ab + bc + ca$. Prove that
$$(a-b)^2 + (b-c)^2 + (c-a)^2 \ge 6(a + b + c - 3).$$
5. Let $0 \le a, b, c \le \frac{1}{2}$ be real numbers with $a + b + c = 1$. Prove that
$$a^3 + b^3 + c^3 + 4abc \le \frac{9}{32}.$$
6. Let $a_1, \ldots, a_n \ge 0$, $n \ge 2$, and $\sum_{i=1}^n a_i = 1$. Find the minimum of
$$\sum_{i=1}^n a_i^2 + 2 \prod_{i=1}^n a_i.$$

9 Extra resources for the curious

- Evan Chen's inequalities handout: `https://web.evanchen.cc/handouts/Ineq/en.pdf`
- Evan Chen's Lagrange multipliers handout: `https://web.evanchen.cc/handouts/LM/LM.pdf`
- For a REALLY GOOD explanation of the intuition behind the multivariable calculus stuff, 3blue1brown's Khan Academy multivariable calculus module: `https://www.khanacademy.org/math/multivariable-calculus/applications-of-multivariable-derivatives`
- Mainly the video on maxima and minima: `https://www.khanacademy.org/math/multivariable-calculus/applications-of-multivariable-derivatives/optimizing-multivariable-functions-videos/v/multivariable-maxima-and-minima`
- Dexter Chua's Optimisation notes, looking at inequalities and finding minima/maxima from a university maths perspective:
`https://dec41.user.srcf.net/notes/IB_E/optimisation.pdf`