Sequences (S)

Jongmin Lim

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Here are some powerful techniques in solving questions with sequences.

1 Linear recurrences

1.1 Linear Homogeneous Recurrences

Example 1: $a_{n+2} = 3a_{n+1} - 2a_n$, where $a_0 = 0$, $a_1 = 1$.
We start with the *ansatz* $a_n = x^n$. This gets us

$$x^{n+2} = 3x^{n+1} - 2x^n$$
$$x^2 - 3x + 2 = 0$$
$$(x-1)(x-2) = 0$$

Hence we conclude that $a_n = 2^n$ and $a_n = 1^n$ are solutions to the recurrence relation. Notice that any linear combination of these two solutions is also a solution to the recurrence. So the general solution to the recurrence is $a_n = A2^n + B1^n$ for some constants A and B. Since $a_0 = A + B = 0$ and $a_1 = 2A + B = 1$, we conclude that $A = 1$, $B = -1$. Hence, the solution to this recurrence is $a_n = 2^n - 1$.
Example 2: $a_{n+2} = 4a_{n+1} - 4a_n$, where $a_0 = 1$, $a_1 = 6$.
Using the same *ansatz*, we get the equation $(x-2)^2 = 0$. Notice that we now have a root with multiplicity. So, alongside the usual solution of $a_n = 2^n$, we also have the solution $a_n = n2^n$ (If it was a triple root, then we have the solution $a_n = n^2 2^n$, etc (check it!)). As before, the general solution is $a_n = (A + Bn)2^n$ for some constants A, B, where we deduce $A = 1$ and $B = 2$ via the initial conditions. Thus, the solution is $a_n = (1 + 2n)2^n$.
Exercises: Find the general term.

1. $a_{n+3} = 6a_{n+2} - 11a_{n+1} + 6a_n$, where $(a_0, a_1, a_2) = (-1, 0, 4)$.
2. $a_{n+3} = a_{n+2} + a_{n+1} - a_n$, where $(a_0, a_1, a_2) = (2, 2, 6)$.
3. $F_{n+2} = F_{n+1} + F_n$, where $(F_0, F_1) = (0, 1)$.

Problems:

1. Find the general term of the fibonacci sequence $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0, F_1 = 1$. Hence show that $F_{n+1}/F_n \to \phi$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.
2. Show that the general term of $a_{n+2} = a_{n+1} - a_n$, where $(a_0, a_1) = (0, 1)$ is $a_n = \frac{2}{\sqrt{3}} \sin\left(\frac{n\pi}{3}\right)$.

3. Let $\{a_n\}$ be a positive sequence such that $a_{n+2} = -a_{n+1} + 2a_n$. Show that a_n must be a constant sequence.

4. Let $\{a_n\}$ be an integer sequence such that $2a_{n+2} = 5a_{n+1} - 3a_n$. Show that a_n must be a constant sequence.

5. Let $\{a_n\}$ be a real valued sequence such that $a_{n+2} - ka_{n+1} + a_n = 0$, for some real number $|k| < 2$. Prove that a_n is bounded.

6. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(f(x)) = -4f(x) + 5x$.

7. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x) > x$ and $f(f(x) - x) = 2x$.

8. (ISL 2010 A5) Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that $f(f(x)^2 y) = x^3 f(xy)$.

9. (China 2017) The sequences $\{u_n\}$ and $\{v_n\}$ are defined by $u_{n+2} = 2u_{n+1} - 3u_n$, where $(u_0, u_1) = (1, 1)$ and $v_{n+3} = v_{n+2} - 3v_{n+1} + 27v_n$, where $(v_0, v_1, v_2) = (a, b, c)$. There exists a positive integer N such that for any $n > N$, u_n divides v_n . Prove that $3a = 2b + c$.

10. (SMMC sample) Consider a circle with circumference $\frac{1+\sqrt{5}}{2}$ and mark a point P_0 on it. Going clockwise, you mark the points $P_1, P_2, \cdots$ for every unit arc length. Suppose you terminate at step n, after you mark the point P_n. Show that if P_i, P_j are adjacent on the circle, then $|i - j|$ is a Fibonacci number.

1.2 Linear Inhomogeneous Recurrences

These are sequences with linear a_n terms, with along with some functions of n. The homogeneous version of the recurrence gives the general part of the solution, while you need a particular solution to take care of the inhomogeneous part (this is where you need to be smart).

Example 1: $a_{n+1} = 3a_n - 2$, where $a_0 = 0$.

The homogeneous version, $a_{n+1} = 3a_n$ has an easy general solution $a_n = A3^n$. We can guess a particular solution to the original equation, which is $a_n \equiv 1$. Thus, the general solution is $a_n = A3^n + 1$, where we tweak $A = -1$ to match the initial condition. Thus, the solution is $a_n = 1 - 3^n$.

The methodical way to do this is to rearrange the original equation to $a_{n+1} - 1 = 3(a_n - 1)$, and define $b_n = a_n - 1$. Then the b_n recurrence is $b_{n+1} = 3b_n$, etc. Sometimes guessing is not that easy, so these substitutions are useful.

Example 2: $a_{n+1} - 5a_n = 5^{n+1}$, where $a_0 = 0$

Divide both sides with 5^{n+1} to get $\frac{a_{n+1}}{5^{n+1}} - \frac{a_n}{5^n} = 1$. Define $b_n = \frac{a_n}{5^n}$, then we get $b_{n+1} - b_n = 1$, with $b_0 = \frac{a_0}{5^0} = 0$. This is yields $b_n = n$, so we conclude $a_n = n5^n$.

Problems:

1. $a_{n+1} - 5a_n = (2n + 1)5^{n+1}$, where $a_0 = 0$.

2. (AMO 2015) Define the sequence $\{a_n\}$ such that $a_{n+2} = 2a_{n+1} - a_n + 2$ with $(a_0, a_1) = (4, 7)$. Show that $a_n a_{n+1}$ is always a term of the sequence.

3. (102 PiC) How many subsets of $\{1, 2, 3, \cdots , 2020\}$ has the property that the sum of its elements is a multiple of 5?

1.3 First Order Multi-variable Homogeneous Recurrences

Sometimes you have more than one sequence working in tandem to define each other.

Example: Find the number of binary sequences of length n such that there does not exist consecutive zeros.

Let a_n, b_n be the number of binary sequences of length n ending with 0 and 1 respectively. As there cannot be consecutive zeros, we have $a_{n+1} = b_n$ (every sequence counted in a_{n+1} is a 0 subtended to a b_n sequence) and $b_{n+1} = a_n + b_n$ (every sequence counted in b_{n+1} is 1 subtended to a a_n or b_n sequence).

Since we're only interested in the sum $x_n := a_n + b_n$, let us inspect this instead.

$$x_n = a_n + b_n = b_{n-1} + b_n = (a_{n-2} + b_{n-2}) + (a_{n-1} + b_{n-1}) = x_{n-2} + x_{n-1}$$

Hence the recurrence is the Fibonacci recurrence.

What if the double-recurrence isn't nice enough so that such manipulations are not immediately obvious?

Our double recurrence can be expressed as $\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$. Let $\mathbf{x_n} := \begin{pmatrix} a_n \\ b_n \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Then $\mathbf{x_{n+1}} = A\mathbf{x_n}$. Thus, our general form becomes $\mathbf{x_n} = A^n\mathbf{x_0}$. The real task is figuring out how to find A^n in a reasonable amount of time. You can *diagonalise* the matrix such that $A = PDP^{-1}$ for some diagonal matrix D (which means we can find D^n easily), so we get $A^n = PD^nP^{-1}$, which is actually solvable (details not included for sanity; look up how to diagonalise a matrix).

1.4 Pell equations

In number theory, this is a Pell equation: $x^2 - dy^2 = 1$ for a non-square number d.

Example: $x^2 - 2y^2 = 1$.

We need an initial solution. There is no method to find this, so good luck with your guesses. In this case, $(x, y) = (3, 2)$ is a solution. Let (x_n, y_n) be a solution to the equation. Then we do a weird factorisation.

$$(x_n - y_n\sqrt{2})(x_n + y_n\sqrt{2}) = 1$$
$$(3 - 2\sqrt{2})(3 + 2\sqrt{2}) = 1$$

Multiplying the terms at their respective locations,

$$((3x_n + 4y_n) - (2x_n + 3y_n)\sqrt{2})((3x_n + 4y_n) + (2x_n + 3y_n)\sqrt{2}) = 1$$

So we see that $(x_{n+1}, y_{n+1}) = (3x_n + 4y_n, 2x_n + 3y_n)$ is a new solution to this Pell equation.

Before you start to write matrices, look a bit closer on how we generated the solutions. $x_n \pm y_n\sqrt{2} = (3 \pm 2\sqrt{2})^n$.

$$x_n = \frac{(3+2\sqrt{2})^n + (3-2\sqrt{2})^n}{2} \qquad y_n = \frac{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}{2\sqrt{2}}$$

Problems:

1. (AMO 2020) Sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_1 = b_1 = 1$ and $a_{n+1} = \frac{a_n+2}{a_n+1}$ and $b_{n+1} = \frac{b_n}{2} + \frac{1}{b_n}$. Show that $b_{n+1} = a_{2^n}$.

2. (Crux) $x_{n+2} = x_{n+1}\sqrt{x_n^2 + 1} + x_n\sqrt{x_{n+1}^2 + 1}$ satisfies $x_0 = x_1 = 1$. Find its general form.

3. (Jongmin) Positive integers x, y satisfy $x^2 = (3y + 4)(y + 1)$. Find all solutions.

2 Telescoping

We want to turn the summation or product into the following forms to simplify nicely.

$$\sum_{k=1}^{n} a_k - a_{k-1} = a_n - a_0$$

$$\prod_{k=1}^{n} \frac{a_k}{a_{k-1}} = \frac{a_n}{a_0}$$

Exercises:

1. Simplify $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{9999\times 10000}$
2. Simplify $\sum_{k=0}^{2017} \frac{k}{(k+1)!}$
3. Simplify $\sum_{k=0}^{2017} \binom{k+8}{k}$
4. Simplify $\prod_{k=1}^{2017} \dfrac{k^2+5k+4}{k^2+5k+6}$
5. Prove that $1.58 < \sum_{n=1}^{\infty} \frac{1}{n^2} < 1.75$

Problems:

1. (Balkan 2016) Find all injective functions $f : \mathbb{R} \to \mathbb{R}$ such that for $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$,
$$\left| \sum_{i=1}^{n} i\left(f(x+i+1) - f(f(x+i))\right) \right| < 2016$$
2. (ISL 2015) Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies
$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$
for every positive integer k. Prove that $a_1 + a_2 + \ldots + a_n \geq n$ for every $n \geq 2$.
3. (USAMTS) Determine the value of
$$S = \sum_{k=1}^{2016} \sqrt{1 + \frac{1}{k^2} + \frac{1}{(k+1)^2}}$$
4. (CMC 2020) Let $a_1, a_2, a_3 \ldots$ be the sequence of positive real numbers such that for each $n \in \mathbb{Z}^+$,
$$\frac{a_1 + a_2 + \cdots a_n}{n} \geq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_{n+1}^2}{n+1}}$$
Show that $\{a_n\}$ is a constant sequence.

3 Convergence

Completeness axiom of real numbers: Every subset $S \subseteq \mathbb{R}$ with an upper bound has a *least upper bound* in $\mathbb{R}$, also known as the supremum s. Note that this axiom does not necessitate that $s \in S$.

Exercises:

1. Let $A = \{n \in \mathbb{Z}^+ \mid 1 - 2^{-n}\}$. What is the supremum s of A? Is s an element of A?

2. Show that every subset $S \subseteq \mathbb{R}$ with a lower bound has a *greatest lower bound* in $\mathbb{R}$, also known as the infimum.

3. Show that this property does not hold for $\mathbb{Q}$. Can you think of an example of a bounded subset of $\mathbb{Q}$ that does not have a supremum in $\mathbb{Q}$?

4. Show that every monotonically increasing sequence with an upper bound converges to a finite limit.

5. Show that every monotonically decreasing sequence with a lower bound converges to a finite limit.

6. Let $a_0 = 2$ and $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$. Show that a_n converges. What is the limit?

Problems:

1. Let $a_n = (-2)^n \sin n\theta$ for some fixed $\theta \neq 0$. Show that for a_n is unbounded.

2. Let $0 \leq f(x) < 1$ denote the fractional part of x; i. e. $f(1.6) = 0.6$, $f(-4.2) = 0.8$. Let $\alpha \in \mathbb{R}$. Show that, for any $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that $0 \leq f(N\alpha) < \epsilon$.

3. (Jongmin) Consider a sequence given by $a_{n+1} = \sqrt{2^{a_n}}$, $a_0 = 1$.

   (a) Show that $\{a_n\}$ converges. What is the limit? Why is the limit not 4?

   (b) For which values of a_0 does $\{a_n\}$ converge? What number does it converge to for each value of a_0?

4. Let $\{a_n\}$ be decreasing positive sequence converging to zero. Define the sequence $\{S_n\}$ by $S_n = \sum_{k=0}^{n} (-1)^k a_n$. Show that $\{S_n\}$ converges.

5. (AMO 1989) u_n satisfies $u_1 = 1$, and $u_1 + u_2 + \ldots u_n = \dfrac{1}{u_{n+1}}$. Determine whether there exist m such that

$$\sum_{i=1}^{m} u_i \geq 1989$$

6. (SMMC 2017) Let $a_1, a_2, a_3 \ldots$ be the sequence of real numbers defined by $a_1 = 1$ and for $m \geq 2$

$$a_m = \frac{1}{a_1^2 + a_2^2 + \cdots + a_{m-1}^2}$$

   Show whether there exist N such that

$$a_1 + a_2 + \cdots + a_N > 2017^{2017}$$

4 Generating functions

Given a sequence $\{a_n\}$, we consider a formal power series $f(x) = a_0 + a_1x + a_2x^2 + \cdots$. We call this the generating function corresponding to the sequence $\{a_n\}$. This is useful as we will see very soon.

4.1 How to get generating functions from sequences

Example 1: What is the generating function of $a_n = 2^n$?
Indeed the generating function is $f(x) = 1 + 2x + 4x^2 + \cdots$. For generating function purposes, we don't really care about the technicalities of convergence. We just use the geometric series formula to simplify $f(x) = \frac{1}{1-2x}$.
Example 2: What is the generating function of $a_n = n$?
Start with the generating function $f(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots$. We can differentiate both sides to obtain

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=0}^{\infty}(n+1)x^n$$

However notice that our terms are off by one (we want $\sum_{n=0}^{\infty} nx^n$). We can fix this easily by multiplying x to both sides. Thus the generating function is $xf'(x) = \frac{x}{(1-x)^2}$.
Example 3: What is the generating function of $a_{n+2} = 3a_{n+1} - 2a_n$ with $a_0 = 2$ and $a_1 = 3$?
Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots$ be the generating function. Then notice

$$\begin{aligned}
f(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_{n+2}x^{n+2} + \cdots \\
xf(x) &= a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \cdots + a_{n+1}x^{n+2} + \\
x^2f(x) &= a_0x^2 + a_1x^3 + a_2x^4 + \cdots + a_nx^{n+2} + \cdots
\end{aligned}$$

According to the recurrence, most of the terms will cancel out in $f(x) - 3xf(x) + 2x^2f(x)$. What terms are left?

$$f(x) - 3xf(x) + 2x^2f(x) = a_0 + a_1x - 3a_0x = 2 - 3x$$

Thus the generating function is $f(x) = \frac{2-3x}{1-3x+2x^2}$. However, notice the following

$$\begin{aligned}
f(x) &= \frac{2-3x}{1-3x+2x^2} \\
&= \frac{2-3x}{(1-x)(1-2x)} \\
&= \frac{1}{1-x} + \frac{1}{1-2x} \\
&= (1 + x + x^2 + \cdots) + (1 + 2x + 4x^2 + \cdots) \\
&= \sum_{n=0}^{\infty}(1+2^n)x^n
\end{aligned}$$

Which is consistent with our understanding of this sequence, because the general term is indeed $a_n = 1 + 2^n$.
Example 4: What is the generating function of $a_{n+1} = \frac{n+2}{n+1}a_n$ with $a_0 = 1$?
Let us try to match the terms.

$$\begin{aligned}
\int_0^x f(t)dt &= a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \cdots + \frac{a_n}{n+1}x^{n+1} + \cdots \\
xf(x) &= a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \cdots + a_nx^{n+1} + \cdots \\
f(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_{n+1}x^{n+1} + \cdots
\end{aligned}$$

Thus

$$\int_0^x f(t)dt + xf(x) = f(x) - a_0$$

Setting $F(x) = \int_0^x f(t)dt$ and $a_0 = 1$

$$\begin{aligned} F + xF' &= F' - 1 \\ F + 1 &= \frac{dF}{dx}(1-x) \\ \frac{dx}{1-x} &= \frac{dF}{F+1} \\ -\ln(1-x) &= \ln(F+1) + C \end{aligned}$$

When $x = 0$, we have $F = 0$ by definition, thus $C = 0$. Therefore

$$F + 1 = \frac{1}{1-x}$$

Thus $F(x) = \frac{x}{1-x}$, so $f(x) = \frac{1}{(1-x)^2}$. Notice that we recognise this from Example 2 to be the generating function for $a_n = n + 1$. Indeed we can see this is the solution to the original sequence.

Exercises:

1. What is the generating function for $a_n = n^2$?

2. What is the generating function for $a_n = n2^n$?

3. What is the generating function for the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$ and $F_0 = F_1 = 1$?

4. What is the generating function for $a_{n+2} = a_n + 2\frac{a_{n+1}}{n+1}$ with $a_0 = 0$ and $a_1 = 67$?

4.2 How to get sequences back from generating functions

Example 1: What is the sequence defined by the generating function $f(x) = e^x$?

We use the Taylor series. The $n!a_n$ would be the constant term after differentiating n times.

$$a_n = \frac{1}{n!}\left[\frac{d^n}{dx^n}f(x)\right]_{x=0} = \frac{1}{n!}\left[e^x\right]_{x=0} = \frac{1}{n!}$$

Thus $f(x)$ defines the sequences $a_n = \frac{1}{n!}$.

Exercises:

1. Show that the sequence given by $f(x) = \frac{1}{\sqrt{1-4x}}$ is $a_n = \binom{2n}{n}$.

2. Show that the sequence given by $f(x) = -\ln(1-x)$ is $a_n = \frac{1}{n}$ for $n \geq 1$ and $a_0 = 0$.

4.3 How to use generating functions to count things

Let a_n as the number of ways to choose n things satisfying some condition, with power series $f(x) = a_0 + a_1x + a_2x^2 + \cdots$. Similarly let b_n be the number of ways to choose n things satisfying another condition with power series $g(x) = b_0 + b_1x + b_2x^2 + \cdots$. One might ask, how many ways can we choose n things that satisfies both conditions? We can choose k things satisfying the first condition, and $n-k$ things satisfying the second condition, and add it all up for $k = 0$ to $k = n$. If we call this c_n, then we have

$$c_n = a_0b_n + a_1b_{n-1} + \cdots a_{n-1}b_1 + a_nb_0$$

Coincidentally, this is exactly what happens to polynomial coefficients when you mutliply them (wow!). Thus our new generating function is $h(x) = c_0 + c_1x + c_2x^2 + \cdots = f(x)g(x)$.

Example 1: You have 1 apples, 2 bananas, 3 carrots. How many ways can you choose 4 things?

The gf for apples is $a(x) = 1+x$, the gf for bananas is $b(x) = 1+x+x^2$ and the gf for carrots is $c(x) = 1+x+x^2+x^3$. The gf for the whole thing is $a(x)b(x)c(x) = (1+x)(1+x+x^2)(1+x+x^2+x^3) = x^6+3x^5+5x^4+6x^3+5x^2+3x+1$. Since the coefficient for x^4 is 5, the answer is 5.

Example 2: How many ways can you choose k people out of a class of n people?

The gf for a single person is $1+x$, since there's one way to choose zero people out of this one person, and one way to choose one person out of this person (weird phrasing I know). Now, the gf for n people is $(1+x)(1+x)\cdots(1+x) = (1+x)^n$. We all know that the coefficient of the n-th term is $\binom{n}{k}$ which is indeed the answer.

Example 3: Let $2a_{n+1} = \frac{1}{n+1}(a_0a_n + a_1a_{n-1} + \cdots + a_{n-1}a_1 + a_na_0)$ with $a_0 = 2$. What is a_{6767}?

Let $f(x)$ be the generating function. Then notice that

$$c_n = a_0a_n + a_1a_{n-1} + \cdots + a_{n-1}a_1 + a_na_0$$

is the n-th term of $f(x)^2$. Thus by comparing coefficients, we have

$$2f'(x) = f(x)^2$$

Solving this, we get $f(x) = \frac{2}{1-x}$, which shows that $a_n = 2$ for all n, so $a_{6767} = 2$.

Example 4: Let the Fibonacci sequence be given by $F_{n+2} = F_{n+1} + F_n$ and $F_0 = F_1 = 1$. What is $\sum_{n=0}^{\infty} F_n2^{-n}$?

When the power series does converge, we can calculate interesting sums. The generating function is

$$F(x) = F_0 + F_1x + F_2x^2 + \cdots = \frac{1}{1-x-x^2}$$

and this indeed converges for $x = \frac{1}{2}$. Therefore subbing in $x = \frac{1}{2}$, we have $\sum_{n=0}^{\infty} F_n2^{-n} = F(\frac{1}{2}) = 4$.

Exercises:

1. You have apples, bananas, carrots, and durians. You have to choose a multiple of 5 apples, even number of bananas, at most 1 carrot and at most 4 durians. How many ways can you choose 2020 things?

2. Let a_n be the number of ways to pay for an n-cent item with $1, 2, 5$ cent coins. What is $\sum_{k=0}^{\infty} a_k3^{-k}$?

3. At a supermarket, there are n different items. How many ways can you choose k things allowing repetition? Use generating functions to show that this is $\binom{n+k-1}{k}$.

Problems:

1. Let $a_n = \binom{2n}{n}$. Construct a recurrence formula for a_n and recover the generating function.

2. Consider c_n to be the n-th Catalan number, which can be characterised as the number of ways to travel from $(0, 0)$ to (n, n) along an integer grid by going right or up, without going above $y = x$. Using this combinatorial description, find a recurrence for c_n and find the generating function of c_n.

3. (102 PiC) How many subsets of $\{1, 2, 3, \cdots, 2020\}$ has the property that the sum of its elements is a multiple of 5?

4. An alternating permutation is a permutation of $\{1, 2, \cdots, n\}$ such that $a_1 < a_2 > a_3 < a_4 > a_5 \cdots$. How many alternating permutations are there?

5. (SMMC 2018) For each positive integer n, consider a cinema with n seats in a row, numbered left to right from 1 up to n. There is a cup holder between any two adjacent seats and there is a cup holder on the right of seat n. So seat 1 is next to one cup holder, while every other seat is next to two cup holders. There are n people, each holding a drink, waiting in line to sit down. In turn, each person chooses an available seat uniformly at random and does the following:

   (a) If they sit next to two empty cup holders, then they choose a cup holder at random and put their drink in it

   (b) If they sit next to one empty cup holder, then they place their drink in that cup holder.

   (c) Otherwise, they get angry.

   Let p_n be the probability that no one gets angry. What is $p_1 + p_2 + p_3 + \cdots$?