

Angle chasing

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1 Basic properties

1. A circle is a set of points with equal distance from a fixed point.
 - (a) Let A, B, C be three points on this circle. Prove that $2\angle BAC = \angle BOC$.
 - (b) In particular, show that the angles on a semicircle is 90° .
 - (c) Let DE be a chord with M as its midpoint. Show that $OM \perp DE$, where O is the centre.
 - (d) Show that if $ABCD$ is a cyclic quadrilateral, then $\angle BAC = \angle BDC$.
 - (e) Show that $\angle ABC + \angle ADC = 180^\circ$.
 - (f) Let AC and BD intersect at X . Show that $AX \times CX = BX \times DX$.
 - (g) Let AD and BC intersect at Y . Show that $YA \times YD = YC \times YB$.
 - (h) Let $T \in BC$ such that AT is a tangent. Show that $\angle TAB = \angle BCA$.
 - (i) Show that $TA^2 = TB \times TC$.
 - (j) Show that (d) to (g) are all if and only if conditions. I.e. if $ABCD$ satisfies those conditions, then $ABCD$ is cyclic.

2 Problems

1. Show that the midpoint of the hypotenuse of a right angled triangle is the circumcentre of the triangle.
2. (Shadow theorem) Let $\triangle ABC$ have $\angle BAC = 90^\circ$ and $D \in BC$ such that $AD \perp BC$. Show that
 - (a) $AB^2 = BD \times BC$
 - (b) $AC^2 = CD \times CB$
 - (c) $AD^2 = BD \times CD$
 - (d) Hence or otherwise prove the pythagorean theorem.
3. Let X be outside circle ω with centre O and let $P, Q \in \omega$ such that PX, QX are tangents. Let $PQ \cap OX = K$. Let $A, B \in \omega$ such that X, A, B is collinear. show that $ABKO$ is cyclic.

4. Let two circles α, β intersect at X, Y . Let $A \in \alpha$ and $B \in \beta$ such that AB is a common tangent. Show that XY meets AB at M , where M is the midpoint of AB .
5. Let A, B be two points on a circle. Let TA, TB be tangents to the circle. Let C be another point on the circle such that $BC \parallel TA$. Let TC meet the circle again at D . Let BD meet TA at M . Show that $TM = MA$.
6. (Spiral sym) Let two circles α, β meet at points X, Y . Let $A, C \in \alpha$ and $B, D \in \beta$ such that AXB and CXD are collinear.
 - (a) Show that $\triangle YAB \sim \triangle YCD$.
 - (b) Show that $\triangle YAC \sim \triangle YBD$.
 - (c) Let M, N be the midpoint of AB and CD . Show that $\triangle YAC \sim \triangle YMN$.
 - (d) Let AC intersect BD at Z . Show that $YAZB$ is cyclic. Show that $YCZD$ is cyclic.
7. (Incentre lemma) Consider D on the midpoint of the arc of BC not containing A . Show that $DB = DC = DI = DJ$, where I is the incentre and J is the excentre of triangle ABC .
8. (Angle bisector lengths) Let D be the midpoint of the arc BC not containing A . Let $AD \cap BC = X$. Show that $DB^2 = DC^2 = DX \times DA$. Also show that $AB \times AC = AX \times AD$.
9. (Shooting lemma) Consider D on the midpoint of the arc of BC not containing A . Choose any two points X, Y on the circle. Let DX, DY meet line BC at points P, Q . Show that $XYPQ$ is cyclic.
10. (Dilation) Let two circles α, β be tangent at point T . Let $A, C \in \alpha$ and $B, D \in \beta$ such that ATB and CTD are collinear. Show that $AC \parallel BD$.
11. (Dilation) Let two circles α, β be tangent at point T . Let $A \in \alpha$ and $B \in \beta$ such that A, T, B are collinear. Let the tangent at A intersect β at X, Y . Show that $BX = BY$.
12. (Nine point circle) Let D, E, F be midpoints of BC, CA, AB of $\triangle ABC$. Let $X \in BC, Y \in CA, Z \in AB$ such that AX, BY, CZ are altitudes of the triangle. Let P, Q, R be midpoints of AH, BH, CH . Show that $DEFXYZPQR$ all lie on one circle.
13. (Simson line) Let $ABCD$ be a cyclic quadrilateral. Let E, F, G be on lines AB, BD, DA such that $AB \perp CE, BD \perp CF, AD \perp CG$. Show that E, F, G are collinear.
14. (2025 Dec prep G6) Triangle ABC has circumcentre O and satisfies $AC = 2AB$. The bisector of $\angle BAC$ intersects side BC at D . Let E be the foot of the perpendicular from O to line AD . Let $F \neq D$ be a point on line AD such that $CD = CF$. Prove that $\angle EBF = \angle ECF$.