

Hard problems

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1 Medium hard problems

1. Let $ABCD$ be a cyclic quadrilateral with diameter AB and centre O . Let $CD \cap AB = X$. Let the circumcircles of $\triangle AOD$ and $\triangle BOC$ meet again at Y . Show that $\angle OYX = 90^\circ$.
2. (APMO 1998) Let ABC be a triangle and $D \in BC$ such that $AD \perp BC$. Let E, D, F be collinear such that $AE \perp BE$, and $AF \perp CF$. Let M be the midpoint of BC , and N be the midpoint of EF . Show that $AN \perp NM$.
3. Let I be the incentre of $\triangle ABC$. Let Γ be the circumcircle of $\triangle ABC$. Let $AI \cap \Gamma = \{A, D\}$. Let E be a point on arc BDC and $F \in BC$ such that $\angle BAF = \angle EAC < \frac{1}{2}\angle BAC$. Let G be the midpoint of IF . Show that $EI \cap DG \in \Gamma$.
4. Let ω be the incircle of $\triangle ABC$. Let ω touch sides BC, CA at D, E respectively. Let D' be the reflection of D over the midpoint of BC . Let E' be the reflection of E over the midpoint of AC . Let $P = AD' \cap BE'$. Let AD' intersect ω at two points, the closer one to A to be labelled Q . Show that $|AQ| = |D'P|$.
5. Let $\triangle ABC$ be inscribed in circle Γ with $AB = AC$. Circles ω_B and ω_C are inscribed in the circular segments given by AB and AC respectively. ω_B and ω_C are tangent to Γ at X, Y respectively. Let the common external tangent to ω_B and ω_C that is closer to A intersect AB, AC at P, Q respectively. Show that PX and QY intersect on the angle bisector of $\angle BAC$.
6. Let $AB + BC = 3AC$ in $\triangle ABC$ with incentre I . Let the incircle touch AB, BC at D, E . Let DK, EL be diametres of the incircle. Show that $ACKL$ is cyclic.
7. Let $\triangle ABC$ have incircle ω which is tangent to BC at D . Let M be the midpoint of altitude AH . Let DM intersect ω again at J . Show that the circumcircle of $\triangle BJC$ is tangent to ω .
8. (USA TST 2011) Let $\triangle ABC$ be an acute triangle with altitudes AD, BE, CF and H as the orthocentre. Let $P, Q \in EF$ such that $AP \perp EF, HQ \perp EF$. Let $DP \cap QH = R$. Show that $HQ = HR$.

9. (2021 Dec prep G7) Let M be the midpoint of side BC of $\triangle ABC$. The circle with centre M and radius AM intersects AB, AC again at P, Q . The tangents to this circle at P, Q intersect at D . Show that the perpendicular bisector of BC bisects AD .
10. (USAMO 2008) Let ABC be an acute, scalene triangle, and let M, N , and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F , inside of triangle ABC . Prove that points A, N, F , and P all lie on one circle.
11. (APMO 2012) Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC , by M the midpoint of BC , and by H the orthocenter of ABC . Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the half line MH , and F be the point of intersection (other than E) of the line ED and the circle Γ . Prove that $\frac{BF}{CF} = \frac{AB}{AC}$ must hold.
12. (APMO 2013) Let $ABCD$ be a quadrilateral inscribed in a circle ω , and let P be a point on the extension of AC such that PB and PD are tangent to ω . The tangent at C intersects PD at Q and the line AD at R . Let E be the second point of intersection between AQ and ω . Prove that B, E, R are collinear.

2 Hard problems

1. (IMO 2024) Let the incenter and incircle of triangle ABC be I and ω , respectively. Let X be the point on line BC different from C such that the line through X parallel to AC is tangent to ω . Similarly, let Y be the point on line BC different from B such that the line through Y parallel to AB is tangent to ω . Let AI intersect the circumcircle of triangle ABC at $P \neq A$. Let K and L be the midpoints of AC and AB , respectively. Prove that $\angle KIL + \angle YPX = 180^\circ$.
2. (Dec camp 2024 G5) Let $ABCD$ be a cyclic quadrilateral inscribed in circle Γ with centre O . Let $AB \cap CD = E$, $BC \cap AD = F$. Let the midpoint of EF be M . Let $T \in \Gamma$ such that MT is a tangent to Γ . Show that $MT = EM$.
3. (IMO 2023) Let ABC be an acute-angled triangle with $AB < AC$. Let Ω be the circumcircle of ABC . Let S be the midpoint of the arc CB of Ω containing A . The perpendicular from A to BC meets BS at D and meets Ω again at $E \neq A$. The line through D parallel to BC meets line BE at L . Denote the circumcircle of triangle BDL by ω . Let ω meet Ω again at $P \neq B$. Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$.
4. (IMO 2008 Q6) Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C , which is also tangent to the lines AD and CD . Prove that the common external tangents to ω_1 and ω_2 intersect on ω .

5. (IMO 2015 G5) Let ABC be a triangle with $CA \neq CB$. Let D , F , and G be the midpoints of the sides AB , AC , and BC respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I , respectively. The points H' and I' are symmetric to H and I about F and G , respectively. The line $H'I'$ meets CD and FG at Q and M , respectively. The line CM meets Γ again at P . Prove that $CQ = QP$.
6. (IMO 2015) Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocenter, and F the foot of the altitude from A . Let M be the midpoint of BC . Let Q be the point on Γ such that $\angle HQA = 90^\circ$ and let K be the point on Γ such that $\angle HKQ = 90^\circ$. Assume that the points A , B , C , K , and Q are all different and lie on Γ in this order. Prove that the circumcircles of triangles KQH and FKM are tangent to each other.
7. (IMO 2025) Let Ω and Γ be circles with centres M and N , respectively, such that the radius of Ω is less than the radius of Γ . Suppose Ω and Γ intersect at two distinct points A and B . Line MN intersects Ω at C and Γ at D , so that C, M, N, D lie on MN in that order. Let P be the circumcentre of triangle ACD . Line AP meets Ω again at $E \neq A$ and meets Γ again at $F \neq A$. Let H be the orthocentre of triangle PMN . Prove that the line through H parallel to AP is tangent to the circumcircle of triangle BEF .
8. (IMO 2014) Points P and Q lie on side BC of acute-angled $\triangle ABC$ so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ , respectively, such that P is the midpoint of AM , and Q is the midpoint of AN . Prove that lines BM and CN intersect on the circumcircle of $\triangle ABC$.
9. (ISL 2016 G2) Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D , E , F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A . Prove that lines XD and AM meet on Γ .
10. (EGMO 2022) Let $ABCD$ be a cyclic quadrilateral with circumcenter O . Let the internal angle bisectors at A and B meet at X , the internal angle bisectors at B and C meet at Y , the internal angle bisectors at C and D meet at Z , and the internal angle bisectors at D and A meet at W . Further, let AC and BD meet at P . Suppose that the points X , Y , Z , W , O , and P are distinct. Prove that O , X , Y , Z , W lie on the same circle if and only if P , X , Y , Z , and W lie on the same circle.
11. (IMO 2006 ISL G9) Points A_1 , B_1 , C_1 are chosen on the sides BC , CA , AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1 , BC_1A_1 , CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2 , B_2 , C_2 respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3 , B_3 , C_3 are symmetric to A_1 , B_1 , C_1 with respect to the midpoints of the sides BC , CA , AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.