Divisibility and Congruences (I)

Jongmin Lim (December 2023)

June 16, 2026

1 Fun factorisations

1.1 Example 1

Find all $n \in \mathbb{Z}^+$ such that $n + 10 | n^3 + 100$

- What else is divisible by $n + 10$ that has the term n^3 in it?

1.2 Example 2

Find all $x, y \in \mathbb{Z}^+$ such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{2022}$

- Can we change the equation into a from of something $\times$ something $=$ number?

1.3 Fun factorisations

Factorise to at least two factors with integer coefficients and nonzero degree.

1. $x^2 - y^2 + 2y - 1$
2. $x^2 - y^2 - 4x + 2y + 3$
3. $2x^2 - xy - x - y^2 + 4y - 3$
4. $a^2b + b^2a + a^2c + c^2a + b^2c + c^2b + 2abc$
5. $a^2b + b^2a + a^2c + c^2a + b^2c + c^2b + 3abc$
6. $a^3 + b^3 + c^3 - 3abc$
7. $x^4 + 4$
8. $x^4 + x^2 + 1$
9. $x^5 + x^4 + 1$
10. $x(x + 1)(x + 2)(x + 3) + 1$

1.4 Problems

1. Find all primes where $p^3 - 4p + 9$ is a perfect square.
2. Find all $n \in \mathbb{Z}^+$ such that $\frac{1}{3} + \frac{1}{n}$ can be expressed as a fraction with a denominator less than n.
3. Find all $n \in \mathbb{Z}$ such that $n^2 + 3n + 1$ divides $n^3 + 6n^2 + 2n + 1$.

4. For which n does x^2+x+1 divide $x^{2n}+x^n+1$?

5. Show that if 4^n+2^n+1 is prime, then n must be a power of 3.

6. Let $P(x)=x^2+x+1$ for positive integers x. Let $Q(x)$ be the largest prime divisor of $P(x)$. Show that $Q(x)$ is never eventually monotonically increasing.

2 Euclidean algorithm

Given $a=bq+r$, we can show that $gcd(a,b)=gcd(b,r)$.

1. Show that $\frac{12n+1}{30n+2}$ is irreducible for all $n \in \mathbb{Z}^+$.

2. Show that two consecutive Fibnacci numbers are always coprime.

3. Find the greatest common divisor between $7^{610}-3^{610}$ and $7^{377}-3^{377}$.

3 GCD trick

3.1 Example 1

Let positive integers a,b,c satisfy $c(ac+1)^2=(5c+2b)(2c+b)$, and c is odd. Show that c must be a perfect square.

1. Let $g=gcd(b,c)$. Why? Because then I can factorise out g and cancel it out. The fact that I take out the gcd also means I get to set $b=gx$ and $c=gy$, and use the fact that $gcd(x,y)=1$.

2. Show that $g|y$.

3. Show that $y|g$. Hence, conclude $y=g$ and solve the problem.

3.2 Example 2

Find square numbers that can be expressed in the form of $t(t+1)$.

1. Notice that t and $t+1$ are coprime. What can we deduce from this?

2. Solve the problem.

3.3 Problems

1. Show that $(36a+b)(36b+a)$ cannot be a power of 2.

2. Find all integer solutions to $y^2(x^2+y^2-2xy-x-y)=(x+y)^2(x-y)$.

3. Find all triples of positive integers a,b,c such that $\frac{(at+1)(bt+1)(ct+1)-1}{lcm(at,bt,ct)}$ is an integer, for any integer t.

4. Let positive integers a,b,c satisfy $c(ac+1)^2=(5c+2b)(2c+b)$, and c is even. Show that there are no solutions.